Contents of the course:

Wealth behaviour under uncertainty

I) Risk aversion

II) Precautionary saving; prudence

III) Portfolio choices with multiple risks: temperance
Methodology

- Primer and notations: the notion of risk aversion
- Theoretical models: model of saving, model of portfolio choice
- Construction of the risk variables
- Econometrics: test of the prediction of the models
Static model: Risk aversion (absolute or relative), risk tolerance

We suppose an agent whose utility function $u(.)$ is assumed to be increasing and concave, and depends only of initial wealth amount $W$:

$$u(W) > 0 \text{ or } < 0 ; \ u' > 0 ; \ u'' < 0$$

$W$ is assumed to be known with certainty. We assume an additive risk $\tilde{y}$ with zero expectation:

$$E\tilde{y}=0$$

Example: a lottery with $y=a$ or $-a$ with probability $1/2$
As utility function \( u \) is concave, the agent supposed to maximise his expected utility, \( Eu(.) \), prefers to take no risk. So, he prefers always to take the expectation of the lottery \( E\tilde{W} \) instead of to participate to the game. In other words, the consumer is risk averse (see Figure 1):

\[
Eu(W + \tilde{y}) < u(W)
\]

With:

\[
E\tilde{W} = E(W + \tilde{y}) = \frac{1}{2}(W+a)+\frac{1}{2}(W-a) = W
\]

So the agent is indifferent between paying the risk premium, \( \Pi \), or participate to the game:
$Eu(W + \tilde{y}) = u(E\tilde{W} - \Pi)$
$= u(W - \Pi) = u(CE\tilde{W})$

with $CE\tilde{W}$ is the certainty equivalent of the game.

Risk premium is the amount that the consumer is willing to pay to avoid risk.

In the same manner, we can define the compensatory risk premium $\Pi^*$ as the following:

$Eu(\tilde{W} + \Pi^*) = Eu(W + \tilde{y} + \Pi^*)$
$= u(E\tilde{W}) = u(W)$

**Figure 1**
Risk aversion

\[ u(W+a) \]
\[ u(W) \]
\[ \text{Eu}(W+\tilde{y}) \]
\[ u(W-a) \]

\( \Pi \)
\( \Pi^* \)

\( W: \) initial wealth ; \( a: \) loss or gain ; \( \Pi: \) risk premium
\( \Pi^*: \) compensatory risk premium
The compensatory risk premium is the amount that we have to pay to individual if we want that he plays.

If the risk is small with small standard deviation $\sigma_y$, we can show (with a Taylor approximation) that this risk premium is at first order (risk aversion in the small, Pratt, 1964):

$$\Pi(W, \tilde{y}) = \Pi^*(W, \tilde{y}) = \left(-\frac{u''}{u'}\right)\sigma_y^2/2 = A\sigma_y^2/2$$

With $A = -\frac{u''}{u'} > 0$

$A$ is absolute risk aversion (Pratt, 1964). Risk tolerance is defined as the inverse of ARA: $\tau = 1/A$. 
It represents the individual's propensity to take risk.

For "big" risk, we can show that a less risk averse consumer (with utility function $u_1$) than another consumer (with utility function $u_2$) will need, every things being equal, a smaller risk premium to play to the lottery (risk aversion in the large):

\[
A_1 < A_2, \forall W \Rightarrow \Pi_1 < \Pi_2, \Pi^*_1 < \Pi^*_2, \forall W
\]

With $u_2$ a positive concave transformation $\rho$ of $u_1$:

\[
\rho(u_1)
\]

So, (Arrow-Pratt theorem) for an agent with decreasing absolute risk aversion (DARA),
$[A'(W) < 0]$, the risk premium decreases according to wealth: $\prod'(W) < 0$.

We can also define relative risk aversion:

$$\gamma(W) = -W \frac{u''}{u'}$$
Application:

Assume a portfolio choice between an risk-less asset and a risky asset.

\[ \tilde{R}_r : \text{return on risky assets} \]

\[ E\tilde{R}_r = \bar{R}_r \quad (\bar{R}_r > R) \]

\[ \sigma_{\tilde{R}_r} = \sigma \text{(small risk)} \]

\[ W : \text{initial wealth} \]

\[ \omega : \text{fraction of wealth invested in risky asset} \]

\[ \text{MaxEu}(\tilde{W}) \]

\[ utc. \quad \tilde{W} = W[R + \omega(\bar{R}_r - R)] \]
We can see that the amount for risky assets $M(.)$ is proportional to the individual's risk tolerance $\tau$.

- Amount invested in risky assets:

\[
M(W) = \frac{(R_r - R)}{\sigma^2 A(W)} \quad A(W) = \text{Absolute risk aversion}
\]

For big risk, we can show that an agent less tolerant than another will choose, every things being equal, a lower amount of risky asset (Arrow, 1965).

\[
\tau_1 < \tau_2, \ \forall \ W \iff A_1 > A_2 \iff M_1(W) < M_2(W)
\]
So $M(W)$ is increasing according to wealth if $u(.)$ is DARA.

We can see also that the proportion of risky assets in wealth is proportional to relative risk aversion $\gamma$. 
Intertemporal model

Precautionary saving, increasing in risk, prudence

Take a consumer planning his choice over two periods, present and future. Capital market is perfect (the individual can borrow over the certain part of his income). The interest rate is assumed to be zero. Resources are assumed to be exogenous (there is no choice on job occupation). The resources of first period are the sum of capital receipts and labour income of first period. \( X \) denotes it and is certain. The second period income
is uncertain and can be expressed by \((Y+\tilde{y})\) where 
\(\tilde{y}\) is an additive risk with zero expectation (see before). \(\tilde{y}\) is exogenous and uninsurable: it's a background risk. \(W\) denotes the amount of resources that are certain. The individual maximises the expectation of an intertemporal utility which is assumed to be additive:

\[
\text{Max}\{C, C'\} \nu(C) + Eu(C')
\]

Under the constraint:

\[
W = X+Y = C+S
\]

\[
C' = S+\tilde{y}
\]
$C$ and $C'$ are consumption of the two periods. $S$ is the amount of "saving". The first order conditions (Euler equation) are:

$$v'(C) = v'(W-S) = Eu'(S+\tilde{y})$$

There will be precautionary saving if $C$ decreases or $S$ increases relatively to the risk-less case (when $\sigma_y = 0$).

$$v'(C) = v'(W-S) = u'(S)$$
As marginal utility $v'(.)$ is decreasing in consumption ($v'$ increases as $S$ increases), the necessary and sufficient condition for precautionary saving is (Leland, 1968, Sandmo, 1970, Drèze and Modigliani, 1972):

$$Eu'(S + \tilde{y}) > u'(S) \iff u' \text{ convex} \iff u'''' > 0$$

This condition is on the third derivative of the utility function. Risk aversion is not sufficient to justify precautionary saving because future consumption is uncertain and so its price is bigger. So uncertainty about future income
implies two effects: an income effect ($C$ decreases - $S$ increases - if $A$ is positive) and a substitution effect ($C$ increases and $S$ decreases).

The previous result is also true for a mean preserving increase in risk, $x$, *a la* Rothschild and Stiglitz (1970): $E(\tilde{x}/\tilde{y}) = 0$. Convexity of marginal utility is the necessary and sufficient condition to an increasing in precautionary saving because we will have:

$$Eu'(S + \tilde{y} + \tilde{x}) > Eu'(S + \tilde{y})$$
This condition implies that \((-u')\) is an increasing and concave function as the utility function \(u\) is. So Kimball (1990) shows that we can apply the results of Arrow-Pratt to \((-u')\) (see Figure 2).

The consumer is "prudent" if \((-u')\) is concave, so:

\[-Eu'(S + \tilde{y}) < -u'(S)\]

With:

\[E\tilde{S} = E(S + \tilde{y}) = 1/2*(S+a)+1/2*(S-a)= S\]
Prudence

S: saving; a: loss or gain; Ψ: precautionary premium
Ψ*: compensatory precautionary premium
As we have defined risk premium with $u$, we can define a *precautionary premium* with $(-u')$, $\Psi$, by:

\[
-Eu'(S + \bar{y}) = -u'(E\bar{S} - \Psi)
\]

\[
= -u'(S - \Psi)
\]

In the same manner, we can define the *compensatory precautionary premium* $\Psi^*$ as:

\[
-Eu'(\tilde{S} + \Psi^*) = -Eu'(S + \bar{y} + \Psi^*)
\]

\[
= -u'(S)
\]
$\Psi^*$ expresses the increasing in saving that compensates (in terms of marginal utility) the introduction of income risk. This is the case of precautionary behaviour (self-insurance behaviour).

If the risk is small with small standard deviation $\sigma_y$, we can show that this precautionary premium is at first order (precautionary saving in the small):

$$\Psi(S, \tilde{y}) = \Psi^*(S, \tilde{y}) = \left[\frac{-(-u')''}{(-u')'}\right]\sigma_y^2/2$$

$$= p\sigma_y^2/2$$

with : $p = \left[\frac{-(-u')''}{(-u')'}\right] = -\frac{u'''}{u''} > 0$
$p$ is absolute prudence. Positive prudence implies precautionary saving. Precautionary premium is, for small risk, proportional to absolute prudence.

For "big" risk, we can show that a less prudent consumer (with minus marginal utility function $-u'_1$) than another consumer (with minus marginal utility function $-u'_2$) will have, every things being equal, a smaller precautionary saving (precautionary saving in the large):

\[
p_1 < p_2, \forall S \Rightarrow \Psi_1 < \Psi_2, \Psi_1 < \Psi_2, \forall S
\]

With $-u'_2$ a positive concave transformation $\rho$ of $-u'_1$:

\[-u'_2 = \rho(-u'_1)\]
So, (Kimball theorem) for an agent with *decreasing absolute prudence (DAP)*, \([p'(S) < 0]\), precautionary decreases according to wealth or consumption: \(\Psi'(W) < 0; \Psi^*(W) < 0\).

We can also define *relative prudence* as we have defined relative risk aversion:

\[
\zeta(C) = -C \frac{u'''}{u''}
\]

There is a perfect analogy between the notion of risk aversion based on \(u\) properties and the notion of prudence based on \(-u'\) properties.
We can show that if \( u \) is DARA, we have:

\[
\frac{A'}{A} = \frac{d\log \frac{-u''}{u'}}{dW} = -\frac{u''}{u'} + \frac{u'''}{u''} = A - p < 0
\]

\( \Leftrightarrow A < p \)

Risk aversion is superior to prudence. In other words, \(-u'\) is more concave or more sensitive to risk than \(u\).
Examples of utility function most often used in models

**Quadratic utility**

\[ u(C) = (-1/2) (C^* - C)^2 : A = 1/(C^* - C) ; p = 0 \]

\( C^* \) is maximum of possible consumption. Absolute risk aversion is increasing according to wealth, which is not realistic. Especially, optimal amount of risky assets is decreasing according to wealth. Absolute prudence is zero, so there is no precautionary saving. For intertemporal model, this felicity function give the case of certainty equivalence.
Exponential utility

\[ u(C) = \frac{-1}{\varepsilon} \exp(-\varepsilon C); \varepsilon > 0 : A = \varepsilon; p = \varepsilon \]

Absolute risk aversion is constant and equal to absolute prudence. Amount of risky assets is independent of wealth. In the intertemporal model, as precautionary do not depend of the amount of wealth, we can distinguish in wealth), life cycle saving (savings intended to finance consumption during old days) and precautionary saving (due to risky income).
**Iso-elastic utility**

\[ u(C) = C^{1-\gamma}/(1 - \gamma) , \gamma > 0 ; \gamma = 1 : u(C) = \log C \]

\[ A = \gamma/C ; \xi = \gamma ; p = (\gamma + 1)/C ; p = \gamma + 1 \]

Individual's preferences are homothetic (when resources increase, consumption of each period increase in the same proportion. Absolute risk aversion and absolute prudence are decreasing. Relative risk aversion and relative prudence are constant. So predictions of theoretical models are more realistic: amount of risky assets are increasing according to wealth; proportion of risky assets is
constant according to wealth. Precautionary saving is decreasing according to wealth.
The definitions of riskiness

⇒ A risk \( \tilde{y} \) is **loss-aggravating** when starting from initial wealth \( W \), if and only if it satisfies

\[
Eu'(W + \tilde{y}) \geq u'(W)
\]

Observe that this is equivalent to:

\[
E\tilde{y} < \Psi
\]
Where $\Psi$ is the precautionary premium as defined by Kimball (1990).

**Demonstration:**

$$Eu'(W + \tilde{y}) = u'(W+E\tilde{y}-\Psi)>u'(W)$$

If $E\tilde{y}=<\Psi \Rightarrow E\tilde{y}-\Psi=<0$

$$\Rightarrow Eu'(W + \tilde{y})=u'(W)$$

The set of risks that satisfy this property for preferences $u$ and initial wealth $W$ is called *expected-marginal-utility-increasing risks*. In intuitive terms, they are risks that make the agent willing to pay a smaller amount than its expected
value in order to keep as optimal the decision prevailing before its introduction. This class of risks are also called non-decreasing expected marginal utility risks.
A risk $\tilde{y}$ is **undesirable** when starting from initial wealth $W$, if and only if it satisfies:

$$Eu(W + \tilde{y}) \leq u(W)$$

This can be restated in terms of the risk premium:

$$E\tilde{y} \leq \Pi$$

Where $\Pi$ is the risk premium as defined by Pratt (1964).
Demonstration:

\[ Eu(W + \tilde{y}) = u(W + E\tilde{y} - \Pi) = <u(W) \]

If \( E\tilde{y} = <\Pi \Rightarrow E\tilde{y} - \Pi = <0 \)

\[ \Rightarrow Eu(W + \tilde{y}) = <u(W) \]

This set of risks are also known as expected-utility-decreasing risks. Intuitively, the agent is willing to pay more than their expected value to take a decision as if he were in a certain environment (according to a certain objective function).
Finally observe that if preferences are DARA, every undesirable risk is loss-aggravating.

If DARA:

\[ p > A \iff \Pi < \Psi \]

\[ \Rightarrow E\tilde{\psi} < \Pi < \Psi \]

So:

\[ \tilde{\psi} \text{ indesirable} \Rightarrow \tilde{\psi} \text{ loss aggravating} \]
A risk $\tilde{y}$ is **unfair** if and only if $E\tilde{y}<0$. As before this can be restated in terms of the risk premium $\Pi$: $E\tilde{y}<0<\Pi$. Observe that undesirable risks have no *a priori* restriction on the sign of their expectation, and thus include unfair risks as a particular class.
Demonstration of standarness (Kimball, 1993)
\[ Eu' (W + \tilde{y}) = u'\left[ W + E\tilde{y} - \psi(W, \tilde{y}) \right] \]

\[ Eu'' (W + \tilde{y}) = \left[ 1 - \frac{\partial\psi(W, \tilde{y})}{\partial W} \right] u'' \left[ W + E\tilde{y} - \psi(W, \tilde{y}) \right] \]

\[ \Rightarrow \frac{-Eu'' (W + \tilde{y})}{Eu' (W + \tilde{y})} = \left[ 1 - \frac{\partial\psi(W, \tilde{y})}{\partial W} \right] A \left[ W + E\tilde{y} - \psi(W, \tilde{y}) \right] \]

\( \psi(.) \) is an increasing function according to \( W \) if DAP, so:
\[
1 - \frac{\partial \psi(W, \tilde{y})}{\partial W} \geq 1
\]

\[
\Rightarrow -\frac{Eu''(W + \tilde{y})}{Eu'(W + \tilde{y})} \geq A[W + E\tilde{y} - \psi(W, \tilde{y})]
\]

if DARA and \( E\tilde{y} \leq 0 \), we have:

\[
A[W + E\tilde{y} - \psi(W, \tilde{y})] < A(W)
\]

\[
\Rightarrow -\frac{Eu''(W + \tilde{y})}{Eu'(W + \tilde{y})} \geq A(W)
\]
The interpretation of the lottery:

"We would like to ask you a hypothetical question that we would like you to answer as if the situation was a real one. You are offered the opportunity of acquiring a security permitting you, with the same probability either to gain 5000 euros or to lose all the capital invested. What is the most (Z) that you are prepared to pay for this security?"

The maximum price for participating in the lottery is then defined by (P is the random price of the lottery):

\[
u(w) = \frac{1}{2} u(w + 5 - Z) + \frac{1}{2} u(w - Z)
= Eu(w + P - Z)
\]

A second Taylor expansion of RHS gives:
Eu\(w + P - Z\) \approx u(w) + u'(w)E(P - Z) \\
+ 0.5u''E(P - Z)^2

\Rightarrow A(w) = \frac{-u''(w)}{u'(w)} = \frac{4(2.5 - Z)}{25 + 2Z^2 - 10Z}